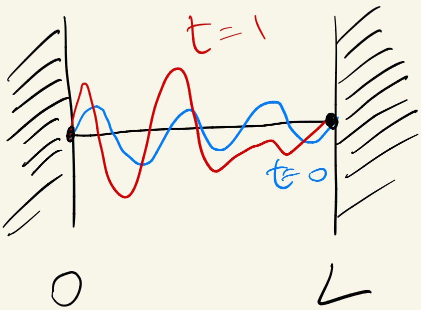
YOU CANNOT HEAR THE SHAPE OF A DRUM

Simpler case: How does a string vibrate?

Ideally:



$u(x,t)$ satisfies the WAVE EQUATION

$$\frac{\partial^2 u}{\partial x^2} = \overset{1}{\overset{\parallel}{c^2}} \frac{\partial^2 u}{\partial t^2}.$$

with fixed velocity $=1$.

with boundary conditions

$$u(0,T) = u(L,T) = 0$$

Simplest solutions: Assume

$$u(x,t) = f(x) \cdot g(t)$$

<u>W.E.</u> : $f''(x)\, g(t) = f(x)\, g''(t)$

$$\Rightarrow \frac{f''(x)}{f(x)} = \frac{g''(t)}{g(t)}$$

$$\Downarrow$$

$$\underbrace{\frac{f''(x)}{f(x)} = -C}, \quad \frac{g''(t)}{g(t)} = -C.$$

$$\Downarrow$$

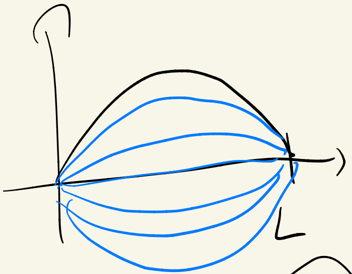
$$\begin{cases} -f''(x) = C f(x) \\ f(0) = f(L) = 0. \end{cases}$$

$\Downarrow$ basic solutions.

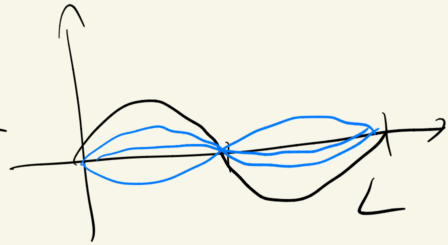
$n \in \mathbb{N}^{>0}$

$$f_n(x) = \sin\left(\frac{n\pi x}{L}\right) \qquad \left(c = \frac{n^2\pi^2}{L}\right)$$

$n=1$



$n=2$



$$g_n(t) = \sin\left(\frac{n\pi t}{L} + b_n\right)$$

Fact the "general" solution is

$$u(x,t) = \sum a_n \sin\left(\frac{n\pi t}{L} + b_n\right) f_n(x).$$

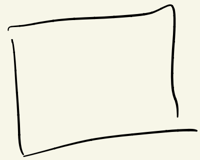
We can thin about it this way:

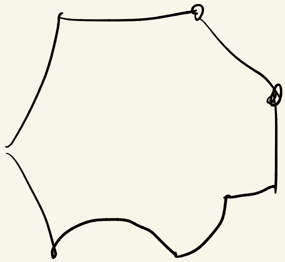
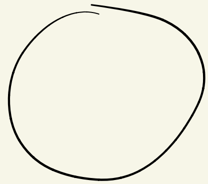
f_n are the eigenvectors of $\boxed{-\frac{\partial^2}{\partial x^2} =}$

$$\begin{cases} -\dfrac{\partial^2}{\partial x^2} f_n = \left(\dfrac{n^2\pi^2}{L^2}\right) f_n \\ f(0) = f(L) = 0. \end{cases}$$

eigenvalue is the square of the frequency.

What about a drum?



$\Omega =$  

$u(x, y, t)$ satisfies.

w. E.
$$\underbrace{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)}_{\Delta \text{ Laplacian}} u = \frac{\partial^2}{\partial t^2} u.$$

$u(x,y) = 0$ for all $x \in \partial\Omega$

$$\left(u|_{\partial\Omega} \equiv 0 \right).$$

The basic modes are solutions

$$f : \Omega \to \mathbb{R}$$

$$\begin{cases} -\Delta f = \lambda f \\ f|_{\partial\Omega} \equiv 0 \end{cases}$$

Dirichlet ∂ condition.

the corresponding eigenvalues

$0 < \lambda_1 \leq \lambda_2 \leq \ldots$ are the squares of the frequencies at which Ω can vibrate.

<u>But all</u> λ_i are very hard to compute in general!

In general

$$\Omega \Rightarrow 0 < \lambda_1 \leq \lambda_2 \leq \ldots$$

Q (Kac '66) is the converse true?

Can you hear the shape of a drum?

Rmk There is more to music than just frequencies.

$\boxed{\text{Spectral geometry}}$

(Thm) λ_i is a discrete set

$\lambda_i \to \infty$.

Thm (Weyl '1911)

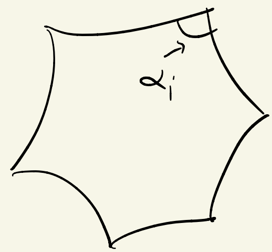
$$\#\{\underbrace{\lambda_n}_{\text{eigenvalues}} \le T\} \underset{\text{asymptotic}}{\sim} \frac{\mathrm{vol}(\Omega)}{4\pi} \cdot T.$$

$\Rightarrow$ you can hear the volume of a drum!

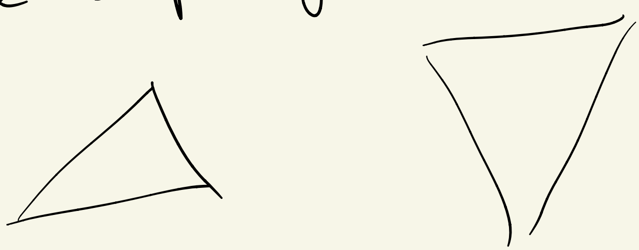
Thm (McKean-Singer '67, Van der Berg '88)

$\{\lambda_n\}$ also determines

- length$(\partial\Omega)$,
- $\sum \frac{1}{\alpha_i}$



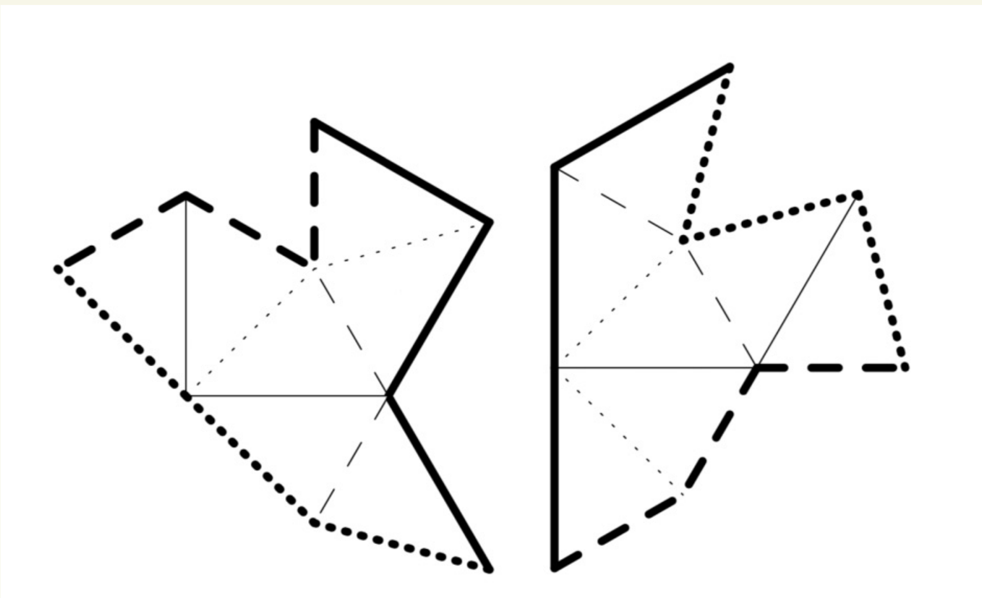
Thm (Durso '90) You can hear the shape of a triangle!

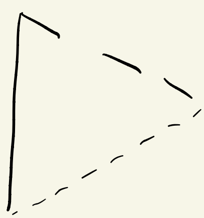


Reference Google hearing the shape of a triangle.

Thm (Gordon, Webb, Wolpert '92)
one cannot hear the shape of a drum.

Example due to Buser, Conway, Doyle, Semmler '94.





← scalene

<u>Claim</u> they are <u>isospectral</u>
(i.e. they sound the same).

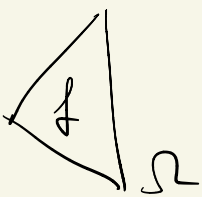
We cannot determine the eigenvalues ...

<u>Idea</u>: "transplant" eigenfunctions

$f : \Omega_0 \to \mathbb{R}$ $\rightsquigarrow$ $\tilde{f} : \Omega_1 \to \mathbb{R}$

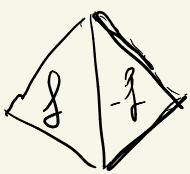
λ-eigenfunction $\quad$ λ-eigenfunction

<u>Lemma</u>



$$\begin{cases} -\Delta f = \lambda f \\ f|_{\partial\Omega} = 0 \end{cases}$$

$\Downarrow$

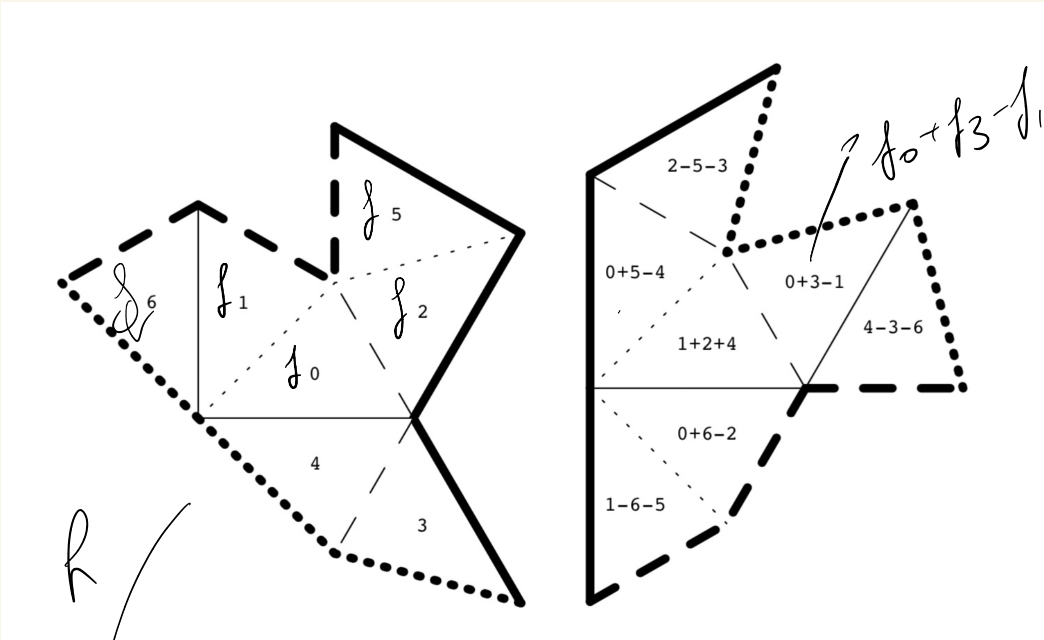


is also λ-eigenfunction!

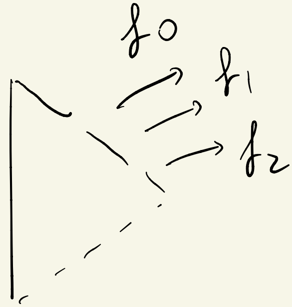
$\longrightarrow \Omega'$

reflecting.

Proof:



$\mathbb{R}$



f:

Open problem are there convex counterexamples?